Top 22 strategies you'll use in Math contests.

Math Gems Festoon

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Contents

§1 AM - GM inequality

The inequality of arithmetic and geometric means (AM–GM inequality) asserts that the geometric mean of a list of non-negative real numbers is less than or equal to the arithmetic mean of the same list and that the two means are only equivalent if and only if each number in the list is the same (in which case they are both that number). In a more compact form, we can write this as

$$\frac{x_1 + x_2 + x_3 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 ... x_n}$$

§1.1 AM-GM inequality proof

The inequality states that if $a_1, a_2, ..., a_n$ whose product is equal to 1, then $a_1 + a_2 + ... + a_n \geq n$, with equality only when $a_1 = a_2 = ... = a_n = 1$ Let us prove this by induction. The case $n = 1$ is trivial; so, let us start with $n \geq 2$. Without the loss of generality, we can assume that a_1 is the maximum ≥ 1 and a_2 is the minimum ≤ 1. Thus we have,

$$a_1 + a_2 - a_1 a_2 - 1 = (a_1 - 1)(1 - a_2) \geq 0$$

i.e.,

$$a_1 + a_2 - a_1 a_2 \geq 1$$

Now, we can also see that

$$a_1 a_2 + a_3 + ... + a_n \geq n - 1$$

Adding the inequalities, we get $a_1 + a_2 + ... + a_n \geq n$

§1.2 Solved examples

1. Show that $\frac{a}{b} + \frac{b}{a} \geq 2$

   Answer: This is a direct application of AM-GM inequality. Setting $\frac{a}{b} = x$ and $\frac{b}{a} = y$, we get that

$$x + y = 2\sqrt{\frac{a}{b} \times \frac{b}{a}} = 2$$

Therefore,

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

2. Proof that

$$(1+a)(1+b)(1+c) \geq 8\sqrt{abc}$$

Answer: Applying the AM-GM inequality in each term, we get

$$(1+a) \geq 2\sqrt{a}$$

$$(1+b) \geq 2\sqrt{b}$$

$$(1+c) \geq 2\sqrt{c}$$

Multiplying them together, we get that

$$(1+a)(1+b)(1+c) \geq 8\sqrt{abc}$$

3. If x,y, and z are positive real numbers satisfying xyz=32, find the minimum value of

$$x^2 + 4xy + 4y^2 + 2z^2$$

Answer: We can write the given expression as

$$x^2 + 2xy + 2xy + 4y^2 + z^2 + z^2$$

Using the AM-GM inequality, we get

$$x^2 + 2xy + 2xy + 4y^2 + z^2 + z^2 \geq 6\sqrt[6]{2^4(xyz)^4} = 96$$

Therefore, the minimum value for the expression is 96, which occurs at $(x, y, z) = (4, 2, 4)$

4. Find the minimum value of

$$f(x) = \frac{9x^2\sin^2(x) + 4}{x\sin(x)}$$

for $0 < x < \pi$

Answer: Let $x\sin(x) = y$. Therefore,

$$f(x) = \frac{9x^2\sin^2(x) + 4}{x\sin(x)} = \frac{9y^2 + 4}{y} \geq 12$$

Therefore, the minimum value of y is $\frac{2}{3}$, and the minimum value of $f(x)$ is 12

5. Let a, b, c be non-negative real numbers. Prove that

$$\sqrt[3]{(1+a)(1+b)(1+c)} \geq 1 + \sqrt[3]{abc}$$

Answer: Using the AM-GM inequality, we get that

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

$$\frac{ab+ac+bc}{3} \geq (\sqrt[3]{abc})^2$$

Adding them together, we get

$$a+b+c+ab+bc+ac \geq 3(\sqrt[3]{abc}+(\sqrt[3]{abc})^2)$$

Adding $1+abc$ to both sides, we get

$$1+a+b+c+ab+bc+ac+abc \geq 1+3\sqrt[3]{abc}+3(\sqrt[3]{abc})^2+abc$$

This becomes

$$(1+a)(1+b)(1+c) \geq (1+\sqrt[3]{abc})^3$$

Taking the cubic root for both sides, we arrive at the requested expression

$$\sqrt[3]{(1+a)(1+b)(1+c)} \geq 1+\sqrt[3]{abc}$$

§2 Cauchy-Shwartz inequality

Cauchy-Shwartz inequality (CS inequality) is one of the most crucial and famous inequalities. In its simplest form, it states that for numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$,

$$(a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_n^2) \geq (a_1b_1+a_2b_2+...+a_nb_n)^2$$

In summation notation, this is written as

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \geq \left(\sum_{i=1}^n a_ib_i\right)^2$$

§2.1 CS-inequality proof

Consider the quadratic trinomial

$$\sum_{i=1}^n (a_ix-b_i)^2 = \sum_{i=1}^n (a_i^2x^2-2a_ib_i+b_i^2)$$

$$= x^2\sum_{i=1}^n a_i^2 - 2x\sum_{i=1}^n a_ib_i + \sum_{i=1}^n b_i^2$$

This trinomial is non-negative for all $x \in \mathbb{R}$, so its discriminant is not positive, i.e.

$$4\left(\sum_{i=1}^n a_ib_i\right)^2 - 4\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \leq 0$$

Therefore, we can conclude that

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \geq \left(\sum_{i=1}^n a_ib_i\right)^2$$

Hence, proven.

§2.2 Solved examples

1. Given that $3a + 4b = 5$, find the minimum value of $a^2 + b^2$

   Answer: Using CS inequality, we have that

   $$(3^2 + 4^2)(a^2 + b^2) \geq (3a + 4b)^2$$

   Therefore,

   $$25(a^2 + b^2) \geq 25$$
   $$= (a^2 + b^2) \geq 1$$

   Hence, the minimum value for $a^2 + b^2$ is 1

2. Given $a_1, a_2, ..., a_n \in \mathbb{R}$ such that $a_1 + a_2 + ... + a_n = 1$, prove that

   $$a_1^2 + a_2^2 + ... + a_n^2 \geq \frac{1}{n}$$

   Answer: Using CS inequality, we get

   $$(a_1^2 + a_2^2 + ... + a_n^2)(n) \geq (1 \times a_1 + 1 \times a_2 + ... + 1 \times a_n)$$

   Therefore, we can clearly see that

   $$a_1^2 + a_2^2 + ... + a_n^2 \geq \frac{1}{n}$$

3. Let $a_1, a_2, \cdots, a_n$, $b_1, b_2, ..., b_n$ be positive real numbers such that $a_1 + a_2 + ... + a_n$ $= b_1 + b_2 + ... + b_n$. Show that

   $$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{a_1 + a_2 + \cdots + a_n}{2}$$

   Answer: The question can be written using summation notation as follows

   $$\sum \frac{a_i^2}{a_i + b_i} \geq \frac{\left(\sum a_i\right)^2}{\sum a_i + \sum b_i}$$

   Moreover, we are given that

   $$\sum a_i + \sum b_i = 2 \sum a_i$$

   Therefore, the RHS will be

   $$\frac{\left(\sum a_i\right)^2}{\sum a_i + \sum b_i} = \frac{\left(\sum a_i\right)^2}{2 \sum a_i} = \frac{\sum a_i}{2}$$

   Therefore,

   $$\sum \frac{a_i^2}{a_i + b_i} \geq \frac{\sum a_i}{2}$$

   This completes the proof.

4. Let a, b, c, d be positive real numbers such that $a+b+c+d=1$. Find the minimum value of
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$
Answer: From CS-inequality, we have
$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 4^2$$
Since $a+b+c+d=1$, we get that
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \geq 16$$
Therefore, the minimum value is 16.

5. For positive real numbers $a, b, c > 0$, show that
$$abc(a+b+c) \leq a^3b+b^3c+c^3a$$
Answer: First, let $(a_1, a_2, a_3) = \left(\frac{a}{\sqrt{c}}, \frac{b}{\sqrt{a}}, \frac{c}{\sqrt{b}}\right)$ and $(b_1, b_2, b_3) = (\sqrt{c}, \sqrt{a}, \sqrt{b})$
Now, using the CS inequality, we get
$$\left(\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}\right)(c+a+b) \geq (a+b+c)^2$$
Therefore, we get that
$$\left(\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}\right) \geq (a+b+c)$$
Multiplying both sides by abc, we get
$$abc(a+b+c) \leq a^3b+b^3c+c^3a$$

§3 Bernoulli's inequality

Bernoulli's inequality is famous for solving inequalities that include exponentiation of $(1+x)$. It states that for $x_1, x_2, ..., x_n$, which are real numbers with the same sign and are greater than -1, we have
$$(1+x_1)(1+x_2)...(1+x_n) \geq 1+x_1+x_2+x_3+...+x_n$$
This inequality has several useful applications and variants.

§3.1 Bernoulli's inequality proof

We will now prove that
$$(1+x)^r \geq 1+rx$$
for every real number $r \geq 1$ and $x \geq -1$ We can see immediately that this is true for $r=0, r=1$. Let us suppose that it is true for $r=k$. Therefore, we have
$$(1+x)^{k+2} = (1+x)^k + (1+x)^2$$

$$\begin{aligned}
&\geq (1+kx)(1+2x+x^2)\\
&= 1+2x+x^2+kx+2kx^2+kx^3\\
&= 1+(k+2)x+kx^2(x+2)+x^2\\
&\geq 1+(k+2)x
\end{aligned}$$

Hence, proven.

§3.2 Solved examples

1. Substitute $x = \frac{5}{n}$ and $x = \frac{-5}{6n}$ to prove that

$$1+\frac{5}{6n-5} \leq \sqrt[n]{6} \leq 1+\frac{5}{n}$$

Answer: 1 - Using Bernoulli's inequality, we know that

$$(1+x)^n \geq 1+nx$$

Using $x = \frac{5}{n}$, we get

$$\left(1+\frac{5}{n}\right)^n \geq 6$$

Taking the nth root, we get

$$1+\frac{5}{n} \geq \sqrt[n]{6}$$

2. Again, we can substitute $x = \frac{-5}{6n}$ and get

$$\left(1-\frac{5}{6n}\right)^n \geq \frac{1}{6}$$

$$\left(\frac{6n-5}{6n}\right)^n \geq \frac{1}{6}$$

Using the reciprocal rule, we get

$$\left(\frac{6n}{6n-5}\right)^n \leq 6$$

After taking the nth root and rewriting the LHS, we get

$$\left(1+\frac{5}{6n-5}\right) \leq \sqrt[n]{6}$$

From (1) and (2), we have that

$$1+\frac{5}{6n-5} \leq \sqrt[n]{6} \leq 1+\frac{5}{n}$$

Hence, proven.

3. Use Bernoulli's equation to show that the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

converges as $n \longrightarrow \infty$

Answer: Let $b_n = a_n(1 + \frac{1}{n})$. We can show that b_n is a decreasing sequence as follows:

$$\frac{b_n}{b_{n-1}} = \frac{(1+n^{-1})^{n+1}}{(1+(n-1)^{-1})^n}$$
$$= \frac{(n^2-1)^n(n+1)}{n^{2n}n}$$
$$\frac{1+n^{-1}}{(1+(n^2-1)^{-1})^n} \leq \frac{1+n^{-1}}{(1+(\frac{n}{n^2-1})}$$
$$< \frac{1+n^{-1}}{1+\frac{n}{n^2}} = 1$$

Therefore, we deduce that $b_{n-1} > b_n$. Since $b_n \geq 1$, b_n converges as $n \longrightarrow \infty$. Consequently, a_n converges as $n \longrightarrow \infty$, as well.

§4 Telescoping sum: How to solve them

A telescoping sum is a sum in which subsequent terms cancel each other, leaving only initial and final terms. A simple example would be

$$S = \sum_{i=1}^{n-1}(a_i - a_{i+1})$$

$$= (a_1 - a_2) + (a_2 - a_3) + ... + (a_{n-1} - a_n) = a_1 - a_n$$

Most of the time, telescoping sums are simplified using partial fraction decomposition.

§4.1 Solved examples

1. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Answer: We can rewrite the question as

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1}$$

$$= \lim_{N\to\infty}\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

$$= \lim_{N\to\infty}\left(1 - \frac{1}{N+1}\right) = 1$$

2. Evaluate

$$\sum_{n=0}^{\infty} \frac{2n+3}{(n+1)(n+2)}$$

Answer:

$$\sum_{n=0}^{\infty} \frac{2n+3}{(n+1)(n+2)} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} \right)$$

$$= \left(1+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)+\ldots+\left(\frac{1}{n-1}+\frac{1}{n}\right)+\left(\frac{1}{n}+\frac{1}{n+1}\right)+\left(\frac{1}{n+1}+\frac{1}{n+2}\right)+\ldots = \infty$$

Therefore, this sum doesn't converge to a single value but diverges.

3. Evaluate

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$$

Answer:

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3} \right)$$

$$\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n-3}-\frac{1}{n-1}\right)+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)+\ldots$$

$$= \frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n-3}$$

As this approach infinity, we get that the sum is equal to $\frac{5}{6}$

§5 Complex numbers

Complex numbers are numbers that extend out the group of the real numbers. It contains the number i defined being equal to $\sqrt{-1}$. They have a wide range of applications in math and physics and can be used as effective problem-solving techniques. Let us define now that a complex number (z) is in the form of $a+bi$, where a, b are real numbers. They adhere to almost all real numbers laws and properties.

An important formula used in many fields, known as Euler's formula, is defined as

$$e^{ix} = \cos x + i \sin x$$

§5.1 Solved examples

1. Find real and/or imaginary solutions to

$$x^2+x+1=0$$

Answer: Using the quadratic formula,

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

we get that the solutions are

$$\left(\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2} \right)$$

2. Express
$$\left(\frac{1+i}{1-i}\right)^3$$
in the form of $a+bi$, where a, b are real numbers.

Answer: Let us start by simplifying $\frac{1+i}{1-i}$. Multiplying by the conjugate $i+1$, we get that
$$\frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{i^2+2i+1}{2} = i$$
Therefore, $i^3 = -i$. So, $a = 0, b = -1$

3. Solve the equation
$$2z^4 - 14z^3 + 33z^2 - 26z + 10 = 0$$
given that one of its roots is $3+i$ where z represents a complex number

Answer: Since $i+3$ is a solution, then $3-i$ is also solution. Therefore, we get that $(i+3)(3-i)$ is a factor of the given equation. Now, we get that
$$2z^2 - 2z + 1 = 0$$
is the other factor in the given equation. Factorizing the equation, we get that
$$(2z-1)^2 = -1$$
Therefore,
$$z = \frac{1}{2} \pm \frac{1}{2}i, 3 \pm i$$

4. Given the equation
$$2z^3 + pz^2 + qz + 16 = 0$$
has solutions α, β, γ, where γ is real. Also, $\alpha = 2(1+i\sqrt{3})$. What is the value of β, γ, p, q

Answer: We can deduce that another solution would be $2(1-i\sqrt{3})$, and since γ is real, we get that this is the value of β. We know, from the multiplication of the roots of a cubic, that
$$\alpha\beta\gamma = \frac{-16}{2} = -8$$
Substituting with the value of α and γ, we get that $\gamma = \frac{-1}{2}$.
The sum of the roots is equal to $\frac{-p}{2}$. Substituting with their values, we get that $p = -7$. We also know that $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{q}{2}$. Therefore, we get that $q = 28$

§6 Take the conjugate!

Taking the conjugate is a simple yet effective problem-solving approach that may be used in a wide range of situations. Mathematically, a conjugate is created by swapping the signs of two terms in a binomial under the assumption that the sum and product of the binomial and its conjugate are rational. In this case, the binomial may be a complex number or a surd. Take a look at the conjugates of the binomials below.

$$2+\sqrt{3} \xrightarrow{\text{Taking conjugate}} 2-\sqrt{3}$$

$$3+5i \xrightarrow{\text{Taking conjugate}} 3-5i$$

When we multiply something with its conjugate we get difference between two squares. Look at the example below:

$$(a+b)(a-b) = a^2 - b^2$$

This property can be very helpful to rationalize the denominator or evaluating sums or even solving inequalities and equations!

Example 6.1 (Evaluating finite sum)

Evaluate the following expression

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots + \frac{1}{\sqrt{2016}+\sqrt{2017}}$$

Solution. All we need to do is to multiply every fraction by its conjugate, for example multiply the first fraction by

$$\frac{1-\sqrt{2}}{1-\sqrt{2}}$$

By doing so, the sum can be simplified as following:

$$\frac{1-\sqrt{2}+\sqrt{2}-\sqrt{3}+\sqrt{3}-\sqrt{4}+\sqrt{4}-\ldots+\sqrt{2016}-\sqrt{2017}}{-1}$$

This expression simplifies to $\frac{1-\sqrt{2017}}{-1}$, which is equal to $\sqrt{2017}-1$ □

Example 6.2 (Solve for x!)

Let a and b be distinct real numbers. Solve the following equation

$$\sqrt{x-b^2} - \sqrt{x-a^2} = a-b$$

Solution. It should be obvious that the following conditions must hold true:

$$x \geq a^2 \quad x \geq b^2$$

Actually the simplest approach to solve this equation is taking the conjugate, another approaches leads to rather complicated computations. Taking the conjugate gives

$$\frac{a^2-b^2}{\sqrt{x-b^2}+\sqrt{x-a^2}} = a-b$$

which is equivalent to

$$\sqrt{x-b^2} + \sqrt{x-a^2} = a+b$$

Adding this to the original equation gives the following:

$$\sqrt{x-b^2} = a$$

This implies that

$$x = \sqrt{a^2+b^2}$$

□

§7 Arithmetic progressions

An arithmetic progression is one of the form

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d, \ldots$$

One important arithmetic sum is

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

To obtain a closed form, we utilise Gauss' trick:
If

$$A_n = 1+2+3+\cdots+n$$

then

$$A_n = n+(n-1)+\cdots+1.$$

Adding these two quantities,

$$\begin{aligned} A_n &= 1+2+\cdots+n \\ A_n &= n+(n-1)+\cdots+1 \\ 2A_n &= (n+1)+(n+1)+\cdots+(n+1) = n(n+1), \end{aligned}$$

since there are n summands. This gives $A_n = \frac{n(n+1)}{2}$, that is,
since there are n summands. This gives $A_n = \frac{n(n+1)}{2}$, that is,

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

For example,

$$1+2+3+\cdots+100 = \frac{100(101)}{2} = 5050.$$

Applying Gauss's trick to the general arithmetic sum

$$(a)+(a+d)+(a+2d)+\cdots+(a+(n-1)d)$$

we obtain

$$(a)+(a+d)+(a+2d)+\cdots+(a+(n-1)d) = \frac{n(2a+(n-1)d)}{2}$$

Example 7.1

Find the sum of all the integers from 1 to 1000 inclusive, which are not multiples of 3 or 5

Solution. One computes the sum of all integers from 1 to 1000 and weeds out the sum of the multiples of 3 and the sum of the multiples of 5 , but puts back the multiples of 15 , which one has counted twice. Put □

$$
\begin{aligned}
A_n &= 1+2+3+\cdots+n, \\
B &= 3+6+9+\cdots+999 = 3A_{333}, \\
C &= 5+10+15+\cdots+1000 = 5A_{200}, \\
D &= 15+30+45+\cdots+990 = 15A_{66}.
\end{aligned}
$$

The desired sum is

$$
\begin{aligned}
A_{1000} - B - C + D &= A_{1000} - 3A_{333} - 5A_{200} + 15A_{66} \\
&= 500500 - 3\cdot 55611 - 5\cdot 20100 + 15\cdot 2211 \\
&= 266332.
\end{aligned}
$$

Example 7.2

Each element of the set $\{10, 11, 12, \ldots, 19, 20\}$ is multiplied by each element of the set $\{21, 22, 23, \ldots, 29, 30\}$.

If all these products are added, what is the resulting sum?

Solution. This is asking for the product $(10+11+\cdots+20)(21+22+\cdots+30)$ after all the terms are multiplied. But

$$10+11+\cdots+20 = \frac{(20+10)(11)}{2} = 165$$

and

$$21+22+\cdots+30 = \frac{(30+21)(10)}{2} = 255.$$

The required total is $(165)(255) = 42075$. □

Example 7.3

The sum of a certain number of consecutive positive integers is 1000 . Find these integers.

Solution. Let the the sum of integers be $S = (l+1)+(l+2)+(l+n)$. Using Gauss' trick we obtain $S = \frac{n(2l+n+1)}{2}$. As $S = 1000, 2000 = n(2l+n+1)$. Now $2000 = n^2+2ln+n > n^2$, whence $n \le \sqrt{2000} = 44$. Moreover, n and $2l+n+1$ divisors of 2000 and are of opposite parity. Since $2000 = 2^4 5^3$, the odd factors of 2000 are $1, 5, 25$, and 125 . We then see that the problem has the following solutions:

$$
\begin{aligned}
n &= 1, l = 999, \\
n &= 5, l = 197, \\
n &= 16, l = 54, \\
n &= 25, l = 27.
\end{aligned}
$$

353 Example Find the sum of all integers between 1 and 100 that leave remainder 2 upon division by 6.

Solution: We want the sum of the integers of the form $6r+2, r = 0, 1, \ldots, 16$. But this is

$$\sum_{r=0}^{16}(6r+2) = 6\sum_{r=0}^{16} r + \sum_{r=0}^{16} 2 = 6\frac{16(17)}{2} + 2(17) = 850$$

□

§8 Sophie Germain Identity

The Sophie Germain Identity states that:

$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$$

One can prove this identity simply by multiplying out the right side and verifying that it equals the left. To derive the factoring, we begin by completing the square and then factor as a difference of squares:

$$\begin{aligned} a^4 + 4b^4 &= a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 \\ &= (a^2 + 2b^2)^2 - (2ab)^2 \\ &= (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab) \end{aligned}$$

Example 8.1

Compute

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}$$

Proof. The Sophie Germain Identity states that a^4+4b^4 can be factored as $\left(a^2+2b^2-2ab\right)\left(a^2+2b^2+2ab\right)$. Each of the terms is in the form of $x^4 + 324$. Using Sophie Germain, we get that

$$\begin{aligned} x^4 + 324 &= x^4 + 4 \cdot 3^4 \\ &= \left(x^2 + 2 \cdot 3^2 - 2 \cdot 3 \cdot x\right)\left(x^2 + 2 \cdot 3^2 + 2 \cdot 3 \cdot x\right) \\ &= (x(x-6) + 18)(x(x+6) + 18), \end{aligned}$$

so the original expression becomes

$$\frac{[(10(10-6)+18)(10(10+6)+18)] \cdots [(58(58-6)+18)(58(58+6)+18)]}{[(4(4-6)+18)(4(4+6)+18)] \cdots [(52(52-6)+18)(52(52+6)+18)]},$$

which simplifies to

$$\frac{(10(4)+18)(10(16)+18)(22(16)+18)(22(28)+18) \cdots (58(52)+18)(58(64)+18)}{(4(-2)+18)(4(10)+18)(16(10)+18)(16(22)+18) \cdots (52(46)+18)(52(58)+18)}.$$

Almost all of the terms cancel out! We are left with $\frac{58(64)+18}{4(-2)+18} = \frac{3730}{10} = \boxed{373}$. □

Example 8.2

Solve in integers. $x - y^4 = 4$ Where x is a prime number.

Proof. Rearrange the equation and use Sophie-Germain to get

$$x = y^4 + 4 = (y^2 + 2y + 2)(y^2 - 2y + 2)$$

Therefore, either $y^2 + 2y + 2 = 1$ or $y^2 - 2y + 2 = 1$. Check the ys that work, and we get that $y = \pm 1$ yields $x = 5, y = \pm 1$, and that is the only solution. $\square$

sectionVieta's formulas Vieta's formulas tell us that the sum and the product of the roots of any polynomial are in the relation to the coefficient of the equation. A quadratic equation which is represented as $f(x) = ax^2 + bx + c$ have roots p and q (the x-values that make the equation equals zero). So, that would be

$$\begin{cases} q + p = \frac{-b}{a} \\ p \times q = \frac{c}{a} \end{cases}$$

Given a three-degree polynomial for example $f(x) = ax^3 + bx^2 + cx + d$, Vieta's formulas help us to find more information about its roots (their sum, their product, and sum of pairwise products), Let r, s, and t are the roots of this polynomial so,

$$\begin{cases} r + t + s = \frac{-b}{a} \\ r \times s \times t = \frac{-d}{a} \\ rs + rt + ts = \frac{c}{a} \end{cases}$$

We will go straight into proof the second and third-degree polynomial, but it also generalizes to higher degrees.

§8.1 Vieta's formulas for quadratic equation proof

$$ax^2 + bx + c = a(x - p)(x - q) \qquad \text{"intercept form"}$$

Now make a distribution,

$$ax^2 + bx + c = a(x - p)(x - q) = a(x^2 - (p + q)x + pq) = ax^2 - a(p + q)x + apq$$

Now compare the coefficient,

$$\begin{cases} ax^2 + bx + c = \\ ax^2 - a(p + q)x + apq \end{cases} \implies b = -a(p + q) \quad \text{and} \quad c = apq$$

We can say that $p + q = \frac{-b}{a}$ and $pq = \frac{c}{a}$.

§8.2 Vieta's formulas for third-degree polynomial proof

$$f(x) = ax^3 + bx^2 + cx + d = k(x - r)(x - t)(x - s) \qquad \text{where } k \text{ is some constant}$$
$$= kx^3 - k(r + s + t)x^2 + k(rs + ts + rt)x - k(rst)$$

Now divide a from both sides as $a = k$, then compare the coefficient,

$$\begin{cases} x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = \\ x^3 - (r + s + t)x^2 + (rs + ts + rt)x - (rst) \end{cases} \implies \begin{cases} -(r + t + s) = \frac{b}{a} \\ (rs + ts + rt) = \frac{c}{a} \\ -(rst) = \frac{d}{a} \end{cases}$$

Hence proven!

Example 8.3

1. Given r, s, and t are 3 distinct roots of $2x^3 + x^2 - 8x + 3$, Find $r^2 + t^2 + s^2$.

Solution:

$$r^2 + t^2 + s^2 = (r + s + t)^2 - 2(rs + rt + ts)$$

Using Vieta's formulas,

$$\begin{cases} r + s + t = -\frac{1}{2} \\ rs + rt + ts = \frac{-8}{2} = -4 \end{cases}$$

So,

$$r^2 + t^2 + s^2 = (-\frac{1}{2})^2 - 2(-4) = 8.25$$

Example 8.4

Given r, s, and t are 3 distinct roots of $x^3 - 2x^2 + 3x - 4$. Find

$$r^2s + s^2r + s^2t + t^2s + t^2r + r^2t$$

Solution:

$$r^2s + s^2r + s^2t + t^2s + t^2r + r^2t = (r + s + t)(rs + tr + st) - 3rts$$

$$\begin{cases} r + s + t = -\frac{-2}{1} = 2 \\ rs + tr + st = \frac{3}{1} = 3 \\ rts = -\frac{-4}{1} = 4 \end{cases}$$

So,

$$r^2s + s^2r + s^2t + t^2s + t^2r + r^2t = 2 \times 3 - 3 \times 4 = -6$$

Example 8.5

If x_1 and x_2 are the roots of the equation $x^2 - 8x + 11 = 0$, determine the value of

$$x_1^3 + x_1^2 + x_1 + x_2^3 + x_2^2 + x_2$$

Solution: First, we will rewrite the expression

$$(x_1^3+x_2^3)(x_1^2+x_2^2)+(x_1+x_2) = (x_1+x_2)(x_1^2+x_2^2-x_1x_2)+(x_1^2+x_2^2+2x_1x_2-2x_1x_2)+(x_1+x_2) =$$

$$(x_1 + x_2)((x_1 + x_2)^2 - 3x_1x_2) + ((x_1 + x_2)^2 - 2x_1x_2) + (x_1 + x_2)$$

Using Vieta's formulas for quadratic equations,

$$\begin{cases} x_1 + x_2 = \frac{-b}{a} = \frac{8}{1} = 8 \\ x_1x_2 = \frac{c}{a} = \frac{11}{1} = 11 \end{cases}$$

$$x_1^3+x_1^2+x_1+x_2^3+x_2^2+x_2 = (x_1+x_2)((x_1+x_2)^2-3x_1x_2)+((x_1+x_2)^2-2x_1x_2)+(x_1+x_2) =$$

$$(8)((8)^2 - 3 \times 11) + ((8)^2 - 2 \times 11) + (8) = 298$$

§9 Functional equations

Functional equations are equations where the unknowns are functions, rather than a traditional variable. However, the methods used to solve functional equations can be quite different than the methods for isolating a traditional variable. We can find the unknown function using (substitution strategy, proving the function is surjective or proving the function is injective).

§9.1 substitution strategy

When encountering functional equations, one of the first things to do is to plug in values. We replace the variables such that x and y with Small numbers for example, $\{0, 1, -1\}$ or with another variables. Look at the following example and you will understand,

Example 9.1

Find all functions s.t,

$$f(\frac{2x-1}{x-3}) = x+3$$

Solution: For this functional equation we can replace $\frac{2x-1}{x-3}$ by another variable for example t and then get the value of x in terms of t,

$$t = \frac{2x-1}{x-3} \implies x = \frac{3t-1}{(t-2)}$$

Now, rewrite the functional equation in terms of t,

$$f(t) = \frac{3t-1}{(t-2)} + 3 = \frac{3t-1}{(t-2)} + \frac{3t-6}{t-2} = \frac{6t-7}{t-2}$$

Finally, we get the functional equation which will be

$$f(x) = \frac{6x-7}{x-2}$$

Another example and you will understand more (but in this example we will substitute with a number):

Example 9.2

Find all functions $f : R \to R$ s.t,

$$f(x + yf(x)) + x = f(xf(y)) + f(y + f(x))$$

Solution: First, we will substitute the variable (or the two) in the equation with a small number say 0,
Replacing x with 0 and also y with 0

$$f(0 + 0f(0)) + 0 = f(0f(0)) + f(0 + f(0)) \implies ff((0)) = 0$$

Now, Replace x with 1 and also y with 1,

$$f(1 + 1f(1)) + 1 = f(1f(1)) + f(1 + f(1)) \implies f(f(1)) = 1$$

Replace x with 1 and y with 0,

$$f(1+0f(1))+1 = f(1f(0))+f(0+f(1)) \implies f(1)+1 = f(f(0))+f(f(1)) \implies f(1) = 0$$

Now we will replace x with 1 and y we will keep it as itself,

$$f(1+yf(1))+1 = f(1f(y))+f(y+f(1)) \implies 1 = f(f(y))+f(y)$$

As we have $f(1) = 0$ and $f(f(1)) = 1$ so, we have that $f(0) = 1$.
Now, replace y with 0 and keep x as itself

$$f(x+0f(x))+x = f(xf(0))+f(0+f(x)) \implies x = f(f(x))$$

From the functional equation that we get before

$$1 = f(f(y))+f(y) \quad \text{we can replace } f(f(y)) \text{ with } y$$

so we get

$$1 = y+f(y) \implies f(y) = 1-y$$

Finally, we get the functional equation which is $f(x) = 1-x$, and then verify your answer does it satisfies the given functional equation or not. If it doesn't satisfy the given functional equation there will be no solutions.

§9.2 surjective functions

Let $f : X \to Y$ be a function. Then f is surjective if every element of Y is the image of at least one element of X. That is, image $(f) = Y$. Symbolically

$$\forall y \in Y \exists x \in X \quad \text{such that} \quad f(x) = y$$

A synonym for "Surjective" is "onto".

For example:

$$f(x) = x \qquad \text{it is surjective on } R$$

$$f(x) = x^2 \qquad \text{it is surjective on } R+ \text{ but it not surjective on } R$$

Look at the following question in functional equations:

Example 9.3

Find all functions $f : R \to R$ s.t,

$$f(f(f(x))+y) = f(f(x-y))+2y$$

Solution: To prove that the function is surjective you will make the left-hand side contain one function of something and the right-hand side will be a surjective quantity. To prove that function is surjective substitute x with y and keep y as itself,

$$f(f(f(y))+y) = f(f(y-y))+2y = f(f(0))+2y$$

so the function is surjective as we have on the right-hand side a surjective quantity (constant "$f(f(0))$" and $2y$).
Now, let's substitute y with 0,

$$f(f(f(x))+0) = f(f(x-0))+2\times 0 \implies f(f(f(x))) = f(f(x))$$

Since our function is surjective so we have that $f(x) = x$ as $f(f(x))$ is also surjective.

§9.3 injective functions

Let $f : X \to Y$ be a function. Then f is injective if distinct elements of X are mapped to distinct elements of Y. That is, if x_1 and x_2 are in X such that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, This is equal to saying if $f(x_1) = f(x_2)$, then $x_1 = x_2$. A synonym for "injective" is "one-to-one function".

Example 9.4

Find all functions $f : R \to R$ s.t,

$$f(f(f(x) + y) + y) = x + y + f(y)$$

Solution: If we substitute y with zero and keep x as itself

$$f(f(f(x) + 0) + 0) = x + 0 + f(0) \implies f \text{ is surjective function}$$

There is a classical way to prove the function is injective,

$$\exists a, b : f(a) = f(b)$$

we want to prove that $a = b$. we will substitute x with a and then we will substitute x with b then we will subtract the two relations to get $a = b$.

$$a + y + f(y) = b + y + f(y) \implies a = b$$

So, the function f is injective. If the function is injective and surjective we call the function "bijective".
Now, replace y with $-x$ and keep x as itself

$$f(f(f(x) - x) - x) = x - x + f(-x) \implies f(f(f(x) - x) - x) = f(-x)$$

As f is injective so we can say that

$$f(f(x) - x) - x = -x \implies f(f(x) - x) = 0$$

Suppose that

$$\exists \alpha : f(\alpha) = 0$$

$$\therefore f(f(x) - x) = 0 = f(\alpha) \implies f(x) - x = \alpha \implies f(x) = \alpha + x$$

Substitute this functional equation in the original question to get the value of α, we will get $\alpha = 0$ So our functional equation is

$$f(x) = x + 0 = x$$

§9.4 Cauchy's functional equations

(First Cauchy equation) Let $f : R \to R$ be a continuous function of a continuous real variable that satisfies the functional relation

$$f(x + y) = f(x) + f(y), \qquad \forall x, y \in R$$

So, $f(x) = cx$ where $c \in R$ is some constant.

(Second Cauchy equation) Let $f : R \to R$ be a continuous function of a continuous real variable that satisfies the functional relation

$$f(x+y) = f(x)f(y), \qquad \forall x, y \in R$$

So, $f(x) = c^x$ where $c \in R+$ is some constant.

(Third Cauchy equation) Let $f : R_+^* \to R$ be a continuous function of a continuous real variable that satisfies the functional relation

$$f(xy) = f(x) + f(y), \qquad \forall x, y \in R_+^*$$

So, $f(x) = c \ln x$ where $c \in R$ is some constant.

(Fourth Cauchy equation) Let $f : R_+ \to R$ be a continuous function of a continuous real variable that satisfies the functional relation

$$f(xy) = f(x)f(y), \qquad \forall x, y \in R_+$$

So, $f(x) = x^n$ where $n \in Z$ is some constant.

§10 Binomial theorem

The binomial theorem (or binomial expansion) is a result of expanding the powers of binomials or sums of two terms. The coefficients of the terms in the expansion are the binomial coefficients $\binom{n}{k}$.

The binomial theorem states that for many positive integers n, we have

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n =$$

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k$$

We can easily prove it by induction.

Example 10.1
What is the coefficient of x^4 in the expansion of $(x+1)^9$?

Solution: The coefficient will be equal to

$$\binom{9}{4} = \frac{9!}{(9-4)!4!} = 126$$

Example 10.2
In the expansion of $(2x + \frac{k}{x})^8$, where k is a positive constant, the term independent of x is 700000. What is the value of k?

Solution: The terms will be in the form

$$\binom{8}{2} \times (2x)^{8-n} \times (\frac{k}{x})^n$$

for the term to be independent of x, We need $(x)^{8-n} \times (\frac{1}{x})^n = x^0 \implies n = 4$. Thus the constant term of

$$\binom{8}{4} \times (2)^4 \times (k)^4 = 700000 \implies k = 5$$

Example 10.3

What is the coefficient of the $x^2y^2z^2$ term in the polynomial expansion of $(x+y+z)^6$?

Solution: In order to achieve a product of $x^2y^2z^2$, the "x" must be "chosen" 2 times out of 6. This gives $\binom{6}{2}$ combinations. Then, the "y" must be "chosen" 2 times out of the remaining 4. This gives $\binom{4}{2}$ combinations. Then, the remaining factors will be "z". Thus, the coefficient of the $x^2y^2z^2$ term is: $\binom{6}{2}\binom{4}{2} = 90$.

§11 Boring absolute values

We are all aware that working with absolute numbers may be tedious when solving equations or inequalities. The first step that most students take when confronted with such problems is to explicitly write the absolute values. Let's take the following easy case in point.

Example 11.1

Solve the equation

$$|2x - 1| = |x + 3|$$

Solution. We have

$$|2x-1| = \begin{cases} -2x+1 & \text{for } x \le \frac{1}{2} \\ 2x-1 & \text{for } x \ge= \frac{1}{2} \end{cases} \quad \text{and} \quad |x+3| = \begin{cases} -x-3 & \text{for } x \le -3 \\ x+3 & \text{for } x \ge -3 \end{cases}$$

If $x \le -3$, the equation becomes $-2x + 1 = -x - 3$; hence $x = 4$. But ¿-3, so we have no solutions in this case. If $-3 < x \le \frac{1}{2}$, we obtain $-2x + 1 = x + 3$ s $x = \frac{-2}{3}$. Finally, if $x > \frac{1}{2}$, we obtain $x = 4$ again. We conclude that the solutions are $x = \frac{-2}{3}$ and $x = 4$ □

Nevertheless, there is an easier way to solve the equation by noticing that $|a| = |b|$ if and only if $a = \pm b$, so that it is not necessary to write the absolute values in an explicit form. Here are some properties of the absolute values that might be useful in solving equations and inequalities:

$$|ab| = |a||b|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

$$|a + b| \le |a| + |b|$$

with equality if and only if $ab \ge 0$

$$|a - b| \le |a| + |b|$$

with equality if and only if $ab \le 0$

Example 11.2

Solve the equation

$$|x-1|+|x-4|=2$$

Solution. Observe

$$|x-1|+|x-4| \geq |(x-1)-(x-4)| = |3| = 3 > 2,$$

hence the equation has no solutions □

§12 Mathematical induction

The desired result is first demonstrated to hold for a specific value (the Base Case), and it is then demonstrated that if the desired result holds for a specific value, it also holds for another, closely related value. This form of evidence is known as induction. Usually, to demonstrate this, one must first demonstrate that the result holds for $n = 1$ (the Base Case) and then demonstrate that the fact that the result holds for $n = k$ indicates that it also holds for $n = k+1$. By demonstrating that the result holds for $n = 1$, which implies that it holds for $n = 1+1 = 2$, which further indicates that it holds for $n = 2+1 = 3$, and so on, we may demonstrate that the result holds for all positive integers.

Other, odder inductions are possible. If a problem asks you to prove something for all integers greater than 3, you can use $n = 4$ as your base case instead. You might have to induct over the even positive integers numbers instead of all of them; in this case, you would take $n = 2$ as your base case, and show that if $n = k$ gives the desired result, so does $n = k+2$. If you wish, you can similarly induct over the powers of 2.

Here is a simple example of how induction works.

Example 12.1

Proof that the sum of the first n integers is equal to $\frac{n(n+1)}{2}$

Proof. Base Case: If $n = 1$, then $1+2+\ldots+n = 1$, and $\frac{1(2)}{2} = 1$. So, $1+2+\ldots+n = \frac{n(n+1)}{2}$ for $n = 1$.

Inductive Step: Suppose the conclusion is valid for $n = k$. That is, suppose we have $1+2+\cdots+k = \frac{k(k+1)}{2}$. Adding $k+1$ to both sides, we get

$$1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2},$$

so the conclusion holding for $n = k$ implies that it holds for $n = k+1$, and our induction is complete. □

Example 12.2 (try by yourself)

A) Prove that the sum of the square of the first n integers is equal to $\frac{n(n+1)(2n+1)}{6}$

B) Prove that

$$\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$$

Example 12.3

If $x + \dfrac{1}{x} = 2\cos x$ so prove that

$$x^n + \frac{1}{x^n} = 2\cos nx$$

Proof. Proof by strong induction:
True for $n = 1$ (given).
If true for $n = 1, 2, 3, ..., n$ then

$$(x^n + \frac{1}{x^n})(x + \frac{1}{x}) = x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}}$$
$$x^{n+1} + \frac{1}{x^{n+1}} = (x^n + \frac{1}{x^n})(x + \frac{1}{x}) - (x^{n-1} + \frac{1}{x^{n-1}})$$
$$= 2\cos nx \cdot 2\cos x - 2\cos(n-1)x$$
$$= 2(\cos(n+1)x + \cos(n-1)x - \cos(n-1)x)$$
$$= 2\cos(n+1)x$$

$\implies$ true for $n + 1$.
Therefore true for all $n \in \mathbb{N}$. □

Example 12.4

Let $\langle a \rangle$ be a sequence satisfying $a_1 = a_2 = 1$ and $a_n = \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-2}})$ for $n > 2$. Prove that $1 \le a_n \le 2$ for n is an element of the naturals.

Proof. Base Cases:

$$a_1 = a_2 = 1 \implies a_1 \in [1, 2] \text{ and } a_2 \in [1, 2]$$

Inductive Step:

$$1 = \frac{1}{2}\left(1 + \frac{2}{2}\right) \le \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-2}}) \le \frac{1}{2}\left(2 + \frac{2}{1}\right) = 2$$
$$\therefore (a_{n-2} \in [1, 2] \text{ and } a_{n-1} \in [1, 2]) \implies a_n \in [1, 2]$$

□

Example 12.5

Prove that:

$$\ln 2 * \ln 3 * \ln 4 * ... * \ln(n+1) \le \frac{(n+1)!}{e^n}; n \in N$$

Proof. We will use induction in order to prove the inequality above.
i) $n = 1$ $\ln 2 \le \frac{2!}{e}$, which is true.
ii) $n = k$ Assume that $\ln 2 \cdot \ln 3 \cdots \ln k \cdot \ln(k+1) \le \frac{(k+1)!}{e^k}$.
iii) $n = k + 1$ Now we are to prove that $\ln 2 \cdot \ln 3 \cdots \ln k \cdot \ln(k+1) \cdot \ln(k+2) \le \frac{(k+2)!}{e^{k+1}}$.
$\ln 2 \cdot \ln 3 \cdots \ln k \cdot \ln(k+1) \cdot \ln(k+2) \le \ln(k+2) \cdot \frac{(k+1)!}{e^k} \le \frac{(k+2)!}{e^{k+1}}$.
We now have to prove that $\ln(k+2) \le \frac{k+2}{e} \implies \ln x \le \frac{x}{e} \implies \ln x^e \le \ln e^x$, which is true.
Hence, the induction is complete. □

§13 Pigeon hole principle

The pigeonhole principle in combinatorics asserts that one hole must have two or more pigeons if $n+1$ or more pigeons are inserted into n holes. This seemingly unimportant assertion may be utilised in incredibly inventive ways to produce powerful arguments for counting; the idea is especially useful in Olympiad contexts.

The theory may also be known as the Dirichlet box theory in earlier writings. The problem with the balls and boxes may be phrased more effectively as follows: If n balls are to be placed in k boxes and $n > k$, then at least one box must contain more than one ball.

Example 13.1

If a Martian has an infinite number of red, blue, yellow, and black socks in a drawer, how many socks must the Martian pull out of the drawer to guarantee he has a pair?

Proof. The Martian must pull 5 socks out of the drawer to guarantee he has a pair. In this case the pigeons are the socks he pulls out and the holes are the colors. Thus, if he pulls out 5 socks, the Pigeonhole Principle states that some two of them have the same color. Also, note that it is possible to pull out 4 socks without obtaining a pair. □

Example 13.2

How many students do you need in a school to guarantee that there are at least 2 students who have the same first two initials?

Proof. There are $26 \times 26 = 676$ different possible sets of two initials that can be obtained using the 26 letters $A, B, C, ..., Z$, so the number of students should be greater than 676. Thus, the minimum number of students is 677. □

Example 13.3

There are 20 students in a school. Any two of them have a grandmother in common. Prove that at least 14 of them have a common grandmother.

Proof. Suppose that there exists a student with only 1 grandmother. It follows that all 20 students have this common grandmother.

Now, let us assume that all students have at least 2 grandmothers. Consider the 2 oldest grandmothers of each student as special.

Among the special grandmothers,

- if there exists 2 distinct special ones, then of course all 20 students will have a common grandmother.

- if there exists 3 distinct special ones, then there will exist at least

$$\left\lceil \frac{20 \times 2}{3} \right\rceil = 14$$

students with the same grandmother. □

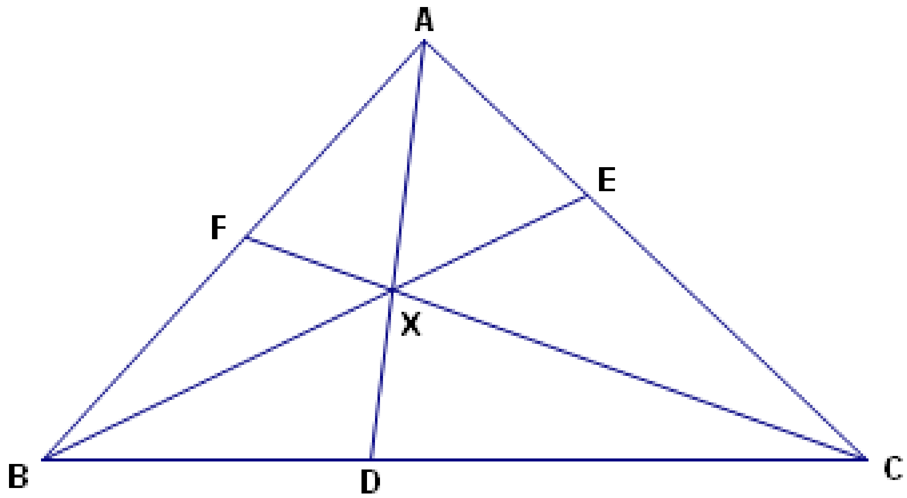
§14 Ceva's theorem

Theorem 14.1

Let ABC be a triangle, and let D, E, F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

This also works for the reciprocal of each of the ratios, as the reciprocal of 1 is 1. (Note that the Cevians do not necessarily lie within the triangle)



Example 14.2

In triangle ABC, $AB = 308$ and $AC = 35$. Given that AD, BE, and CF, intersect at P and are an angle bisector, median, and altitude of the triangle, respectively, compute the length of BC.

Proof. Let $BC = x$.

By the Angle Bisector Theorem,

$$\frac{CD}{BD} = \frac{AC}{AB} = \frac{35}{308} = \frac{5}{44}$$

Let $CF = h$. Then by the Pythagorean Theorem, $h^2 + AF^2 = 35^2$ and $h^2 + BF^2 = x^2$. Subtracting the former equation from the latter to eliminate h, we have $BF^2 - AF^2 = x^2 - 35^2$ so $(BF + AF)(BF - AF) = x^2 - 1225$.
Since $BF + AF = AB = 308$, $BF - AF = \frac{x^2 - 1225}{308}$.
We can solve these equations for BF and AF in terms of x to find that

$$BF = 154 + \frac{x^2 - 1225}{616} =$$

and

$$AF = 154 - \frac{x^2 - 1225}{616}$$

Now, by Ceva's Theorem,

$$\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = 1$$

, so $1 \cdot \frac{5}{44} \cdot \frac{BF}{AF} = 1$ and $5BF = 44AF$. Plugging in the values we previously found,

$$5 \cdot 154 + \frac{5(x^2 - 1225)}{616} = 44 \cdot 154 - \frac{44(x^2 - 1225)}{616}$$

so

$$\frac{49}{616}(x^2 - 1225) = 39 \cdot 154$$

and

$$x^2 - 1225 = 75504$$

which yields finally $x = 277$. □

§15 Product rule

(Product Rule) Suppose that an experiment E can be performed in k stages: E_1 first, E_2 second, ..., E_k last. Suppose moreover that E_i can be done in n_i different ways, and that the number of ways of performing E_i is not influenced by any predecessors $E_1, E_2, \ldots, E_{i-1}$. Then E_1 and E_2 and ... and E_k can occur simultaneously in $n_1 n_2 \cdots n_k$ ways.

Example 15.1

In a group of 8 men and 9 women we can pick one man and one woman in $8 \cdot 9 = 72$ ways. Notice that we are choosing two persons.

Example 15.2

A red die and a blue die are tossed. In how many ways can they land?

If we view the outcomes as an ordered pair (r, b) then by the multiplication principle we have the $6 \cdot 6 = 36$ possible outcomes

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

So, the red die can land in 6 ways and also the blue die may land in 6 ways.

Example 15.3

A multiple-choice test consists of 20 questions, each one with 4 choices. There are 4 ways of answering the first question, 4 ways of answering the second question, etc., hence there are $4^{20} = 1099511627776$ ways of answering the exam.

Example 15.4

There are $9 \cdot 10 \cdot 10 = 900$ positive 3-digit integers:

$$100, 101, 102, \ldots, 998, 999.$$

For, the leftmost integer cannot be 0 and so there are only 9 choices $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for it,

$$9, x, y$$

There are 10 choices for the second digit
and also 10 choices for the last digit

$$9, 10, y$$

Remark 15.5. Definition A palindromic integer or palindrome is a positive integer whose decimal expansion is symmetric and that is not divisible by 10 . In other words, one reads the same integer backwards or forwards. [1]

Example 15.6

The following integers are all palindromes:

$$1, 8, 11, 99, 101, 131, 999, 1234321, 9987899.$$

Example 15.7

How many palindromes are there of 5 digits?

Solution. □

There are 9 ways of choosing the leftmost digit.

$9, a, b, c, d$ Once the leftmost digit is chosen, the last digit must be identical to it, so we have

$9, a, b, c, 1$ There are 10 choices for the second digit from the left $9, 10, b, c, 1$ Once this digit is chosen, the second digit from the right must be identical to it, so we have only 1 choice for it, $9, 10, b, 1, 1$ Finally, there are 10 choices for the third digit from the right,

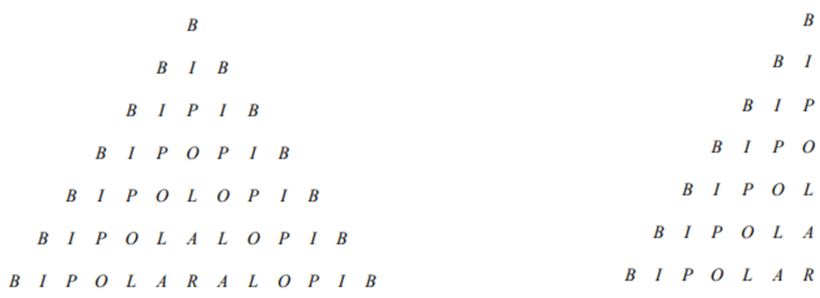
$9, 10, 10, 1, 1$ which give us 900 palindromes of 5-digits.

Example 15.8

How many positive divisors does 300 have? Solution: We have $300 = 3 \cdot 2^2 5^2$. Thus every factor of 300 is of the form $3^a 2^b 5^c$, where $0 \leq a \leq 1, 0 \leq b \leq 2$, and $0 \leq c \leq 2$. Thus there are 2 choices for $a, 3$ for b and 3 for c. This gives $2 \cdot 3 \cdot 3 = 18$ positive divisors.

Example 15.9

How many paths consisting of a sequence of horizontal and/or vertical line segments, each segment connecting a pair of adjacent letters in figure 5.6 spell BIPOLAR?



Solution. Split the diagram. Since every required path must use the R, we count paths starting from R and reaching up to a B. Since there are six more rows that we can travel to, and since at each stage we can go either up or left, we have $2^6 = 64$ paths. The other half of the figure will provide 64 more paths. Since the middle column is shared by both halves, we have a total of $64 + 64 - 1 = 127$ paths.

$\square$

We now prove that if a set A has n elements, then it has 2^n subsets. To motivate the proof, consider the set $\{a, b, c\}$. To each element we attach a binary code of length 3 . We write 0 if a particular element is not in the set and 1 if it is. We then have the following associations:

$$\varnothing \leftrightarrow 000,$$
$$\{a\} \leftrightarrow 100,$$
$$\{b\} \leftrightarrow 010,$$
$$\{c\} \leftrightarrow 001,$$
$$\{a, b\} \leftrightarrow 110,$$
$$\{a, c\} \leftrightarrow 101,$$
$$\{b, c\} \leftrightarrow 011,$$
$$\{a, b, c\} \leftrightarrow 111.$$

Thus there is a one-to-one correspondence between the subsets of a finite set of 3 elements and binary sequences of length 3 . 452 Problem Out of nine different pairs of shoes, in how many ways could I choose a right shoe and a left shoe, which should not form a pair?

§15.1 The Sum Rule

(Sum Rule: Disjunctive Form) Let $E_1, E_2, \ldots, E_k$, be pairwise mutually exclusive events. If E_i can occur in n_i ways, then either E_1 or E_2 or, $\ldots$, or E_k can occur number of ways equals to:

$$n_1 + n_2 + \cdots n_k$$

Notice that the "or" here is exclusive.

Example 15.10

In a group of 8 men and 9 women we can pick one man or one woman in $8 + 9 = 17$ ways. Notice that we are choosing one person.

Example 15.11

There are five Golden retrievers, six Irish setters, and eight Poodles at the pound. How many ways can two dogs be chosen if they are not the same kind.

Solution. We choose: a Golden retriever and an Irish setter or a Golden retriever and a Poodle or an Irish setter and a Poodle.

One Golden retriever and one Irish setter can be chosen in $5 \cdot 6 = 30$ ways; one Golden retriever and one Poodle can be chosen in $5 \cdot 8 = 40$ ways; one Irish setter and one Poodle can be chosen in $6 \cdot 8 = 48$ ways. By the sum rule, there are $30 + 40 + 48 = 118$ combinations.

□

Example 15.12

To write a book 1890 digits were utilised. How many pages does the book have?

Solution. A total of

$$1 \cdot 9 + 2 \cdot 90 = 189$$

digits are used to write pages 1 to 99 , inclusive. We have of $1890 - 189 = 1701$ digits at our disposition which is enough for $1701/3 = 567$ extra pages (starting from page 100). The book has $99 + 567 = 666$ pages.

□

Example 15.13

The sequence of palindromes, starting with 1 is written in ascending order

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, \ldots$$

Find the 1984-th positive palindrome.

Solution. It is easy to see that there are 9 palindromes of 1-digit, 9 palindromes with 2-digits, 90 with 3-digits, 90 with 4-digits, 900 with 5-digits and 900 with 6-digits. The last palindrome with 6 digits, 999999 , constitutes the $9 + 9 + 90 + 90 + 900 + 900 = 1998$ th palindrome. Hence, the 1997th palindrome is 998899 , the 1996th palindrome is 997799 , the 1995th palindrome is 996699 , the 1994th is 995599 , etc., until we find the 1984th palindrome to be 985589. □

Example 15.14

The integers from 1 to 1000 are written in succession. Find the sum of all the digits.

Example 15.15

When writing the integers from 000 to 999 (with three digits), $3 \times 1000 = 3000$ digits are used. Each of the 10 digits is used an equal number of times, so each digit is used 300 times. The the sum of the digits in the interval 000 to 999 is thus

$$(0+1+2+3+4+5+6+7+8+9)(300) = 13500.$$

Therefore, the sum of the digits when writing the integers from 000 to 1000 is $13500 + 1 = 13501$.

Example 15.16

How many 4-digit integers can be formed with the set of digits $\{0, 1, 2, 3, 4, 5\}$ such that no digit is repeated and the resulting integer is a multiple of 3 ?

Solution. The integers desired have the form $D_1D_2D_3D_4$ with $D_1 \neq 0$. Under the stipulated constraints, we must have

$$D_1 + D_2 + D_3 + D_4 \in \{6, 9, 12\}.$$

We thus consider three cases.

Case I: $D_1+D_2+D_3+D_4 = 6$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 1, 2, 3, 4\}, D_1 \neq 0$. There are then 3 choices for D_1. After D_1 is chosen, D_2 can be chosen in 3 ways, D_3 in 2 ways, and D_1 in 1 way. There are thus $3 \times 3 \times 2 \times 1 = 3 \cdot 3! = 18$ integers satisfying case I.

Case II: $D_1 + D_2 + D_3 + D_4 = 9$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 2, 3, 4\}, D_1 \neq 0$ or $\{D_1, D_2, D_3, D_4\} = \{0, 1, 3, 5\}, D_1 \neq 0$. Like before, there are $3 \cdot 3! = 18$ numbers in each possibility, thus we have $2 \times 18 = 36$ numbers in case II.

Case III: $D_1+D_2+D_3+D_4 = 12$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 3, 4, 5\}, D_1 \neq 0$ or $\{D_1, D_2, D_3, D_4\} = \{1, 2, 4, 5\}$. In the first possibility there are $3 \cdot 3! = 18$ numbers, and in the second there are $4! = 24$. Thus we have $18 + 24 = 42$ numbers in case III.

The desired number is finally $18 + 36 + 42 = 96$. □

§16 Permutations

In combinatorics, a permutation is an ordering of a list of objects. Permutations are important in a variety of counting problems (particularly those in which order is important), as well as in various other areas of mathematics; for example, the determinant is often defined using permutations.

The number of permutations of n things taken k at a time is

$$P(n, k) = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

A permutation of some objects is a particular linear ordering of the objects; $P(n, k)$ in effect counts two things simultaneously: the number of ways to choose and order k out of n objects. A useful special case is $k = n$, in which we are simply counting the number of ways to order all n objects. This is $n(n-1)\cdots(n-n+1) = n!$. If we use the another form of permutation

$$\frac{n!}{(n-k)!} = \frac{n!}{(n-n)!} = \frac{n!}{(0)!}$$

We can conclude an important result that $0! = 1$.

Example 16.1

How many outcomes are possible when three dice are rolled, if no two of them may be the same?

Solution:

$$6P3 = \frac{6!}{(6-3)!} = 6 \times 5 \times 4 = 120$$

Example 16.2

How many arrangements can be made out of the letters of the word

"PERMUTATIONS"

?

Solution: As we have in this word 2 letters of "T" we will divide in the end by 2! (permutation with repetition) as it appeared twice

$$\frac{12!}{2!} = 239500800$$

Example 16.3

Lisa has 4 different dog ornaments and 6 different cat ornaments that she wants to place on her mantle. All of the dog ornaments should be consecutive and the cat ornaments should also be consecutive. How many ways can they be arranged?

Solution: We have to decide if we want to place the dog ornaments first, or the cat ornaments first, which gives us 2 possibilities. We can arrange the cat ornaments in 6! ways and the dog ornaments in 4! ways. Hence, by the rule of product, there are $2 \times 6! \times 4! = 34560$ ways to arrange the ornaments (Permutations with Restriction).

§17 Combination

A combination is a way of choosing elements from a set in which order does not matter. A wide variety of counting problems can be cast in terms of the simple concept of combinations.

The number of subsets of size k of a set of size n (also called an $n-$set) is

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

The notation $C(n,k)$ is rarely used; instead, we use $\binom{n}{k}$, pronounced "n choose k".

Example 17.1

How many ways are there to select 3 males and 2 females out of 7 males and 5 females?

Solution:

$$\frac{7!}{3!(7-3)!} \times \frac{5!}{2!(5-2)!} = 350$$

Example 17.2

How many ordered non-negative integer solutions (a, b, c, d) are there to the equation $a + b + c + d = 10$?

Solution: In this question, we can the formula of "stars and bars" which was popularized by William Feller

$$^{n-r+1}C_n =^{10+4-1} C_{10} = \binom{13}{10} = \frac{13!}{10!(13-10)!} = 286$$

Example 17.3

There are two distinct boxes, 10 identical red balls, 10 identical yellow balls, and 10 identical blue balls. How many ways are there to sort the 30 balls into the two boxes so that each box has 15?

Solution: Let a, b, and c be the number of red, yellow, and blue balls, respectively. We need to make $a + b + c = 15$, and then subtract the number of the cases where $a > 10$, $b > 10$ or $c > 10$. to find the number of cases satisfying that $a + b + c = 15$ we will use the formula of stars and bars which is

$$^{n-r+1}C_n =^{17} C_{15} = 136$$

Now, the number of cases where $10 < r \leq 15$ is $1 + 2 + 3 + 4 + 5 = \frac{5\times 6}{2} = 15$. Since exactly the same applies for $10 < y \leq 15$ or $10 < c \leq 15$, the total number of cases that we will subtract it from the total cases 136 is

$$15 \times 3 = 45$$

The answer is $136 - 45 = 91$.

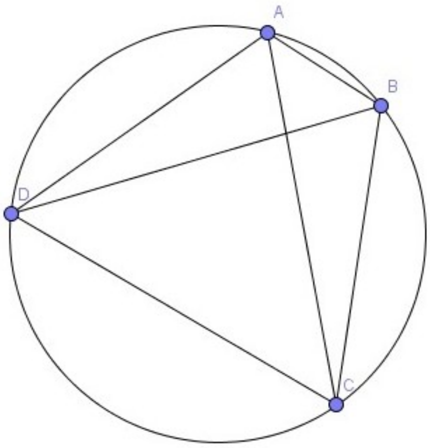
§18 Ptolemy's theorem

Ptolemy's theorem states the relationship between the diagonals and the sides of a cyclic quadrilateral. It is a powerful tool to apply to problems about inscribed quadrilaterals.

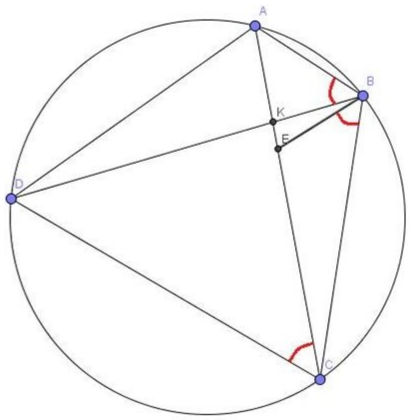
Theorem 18.1

If a quadrilateral is inscribed in a circle, then the product of the measures of its diagonals is equal to the sum of the products of the measures of the pairs of the opposite sides:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$



Proof. Let $ABCD$ be a random quadrilateral inscribed in a circle.



The proposition will be proved if $AC \cdot BD = AB \cdot CD + AD \cdot BC$
It's easy to see in the inscribed angles that $\measuredangle ABD = \measuredangle ACD$, $\measuredangle BDA = \measuredangle BCA$, $\measuredangle BCA = \measuredangle BDA$

$$\Delta EBC \approx \Delta ABD \iff \frac{CB}{DB} = \frac{CE}{AD} \iff AD \cdot CB = DB \cdot CE \quad \textit{[1]}$$

Note that $\measuredangle ABD = \measuredangle EBC \iff \measuredangle ABD + \measuredangle KBE = \measuredangle EBC + \measuredangle KBE \longrightarrow \measuredangle ABE = \measuredangle CBk$, then since $\measuredangle ABE = \measuredangle CBK$ and $\measuredangle CAB = \measuredangle CDB$

$$\Delta ABE = \Delta BDC \iff \frac{AB}{DB} = \frac{AE}{CD} \iff CD \cdot AB = DB \cdot AE \quad \textit{[2]}$$

Therefore from *[1]*, *[2]*, we have

$$\begin{aligned} AB \cdot CD + AD \cdot BC &= CE \cdot DB + AE \cdot DB \\ &= (CE + AE)DB \\ &= CA \cdot DB \end{aligned}$$

Hence proved. □

§19 Geometric Sums

A geometric progression is one of the form

$$a, ar, ar^2, ar^3, \dots, ar^{n-1}, \dots,$$

Example 19.1
Find the following geometric sum:

$$1 + 2 + 4 + \cdots + 1024.$$

Solution. Let

$$S = 1 + 2 + 4 + \cdots + 1024.$$

Then

$$2S = 2 + 4 + 8 + \cdots + 1024 + 2048.$$

Hence

$$S = 2S - S = (2 + 4 + 8 \cdots + 2048) - (1 + 2 + 4 + \cdots + 1024) = 2048 - 1 = 2047$$

□

Example 19.2
Find the geometric sum

$$x = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{99}}.$$

Solution. We have

$$\frac{1}{3}x = \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{99}} + \frac{1}{3^{100}}.$$

Then

$$\begin{aligned} \frac{2}{3}x =& x - \frac{1}{3}x \\ =& \left(\frac{1}{3} + \frac{1}{3^7} + \frac{1}{3^3} + \cdots + \frac{1}{3^{99}}\right) \\ & \qquad \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{99}} + \frac{1}{3^{100}}\right) \\ =& \frac{1}{3} - \frac{1}{3^{100}}. \end{aligned}$$

From which we gather

$$x = \frac{1}{2} - \frac{1}{2 \cdot 3^{99}}.$$

□

The following example presents an arithmetic-geometric sum.

Example 19.3

sum

$$a = 1 + 2 \cdot 4 + 3 \cdot 4^2 + \cdots + 10 \cdot 4^9.$$

Solution. We have

$$4a = 4 + 2 \cdot 4^2 + 3 \cdot 4^3 + \cdots + 9 \cdot 4^9 + 10 \cdot 4^{10}.$$

Now, $4a - a$ yields

$$3a = -1 - 4 - 4^2 - 4^3 - \cdots - 4^9 + 10 \cdot 4^{10}.$$

Adding this last geometric series,

$$a = \frac{10 \cdot 4^{10}}{3} - \frac{4^{10} - 1}{9}.$$

□

Example 19.4

$$S_n = 1 + 1/2 + 1/4 + \cdots + 1/2^n.$$

Interpret your result as $n \to \infty$.

Solution. We have

$$S_n - \frac{1}{2}S_n = (1 + 1/2 + 1/4 + \cdots + 1/2^n) - \big(1/2 + 1/4 + \cdots + 1/2^n + 1/2^{n+1}\big) = 1 - 1/2^n.$$

hence

$$S_n = 2 - 1/2^n.$$

So as n varies, we have:

$$\begin{aligned}
S_1 &= 2 - 1/2^0 = 1 \\
S_2 &= 2 - 1/2 = 1.5 \\
S_3 &= 2 - 1/2^2 = 1.875 \\
S_4 &= 2 - 1/2^3 = 1.875 \\
S_5 &= 2 - 1/2^4 = 1.9375 \\
S_6 &= 2 - 1/2^5 = 1.96875 \\
S_{10} &= 2 - 1/2^9 = 1.998046875
\end{aligned}$$

Thus the farther we go in the series, the closer we get to 2 . Let us sum now the geometric series

$$S = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Plainly, if $r = 1$ then $S = na$, so we may assume that $r \neq 1$. We have

$$rS = ar + ar^2 + \cdots + ar^n.$$

Hence

$$S - rS = a + ar + ar^2 + \cdots + ar^{n-1} - ar - ar^2 - \cdots - ar^n = a - ar^n.$$

From this we deduce that

$$S = \frac{a - ar^n}{1 - r},$$

that is,

$$a + ar + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}$$

If $|r| < 1$ then $r^n \to 0$ as $n \to \infty$.
For $|r| < 1$, we obtain the sum of the infinite geometric series

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r}$$

$\square$

Example 19.5

A fly starts at the origin and goes 1 unit up, 1/2 unit right, 1/4 unit down, 1/8 unit left, 1/16 unit up, etc., ad infinitum. In what coordinates does it end up?

Solution: Its x coordinate is

$$\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \cdots = \frac{\frac{1}{2}}{1 - \frac{-1}{4}} = \frac{2}{5}.$$

Its y coordinate is

$$1 - \frac{1}{4} + \frac{1}{16} - \cdots = \frac{1}{1 - \frac{-1}{4}} = \frac{4}{5}.$$

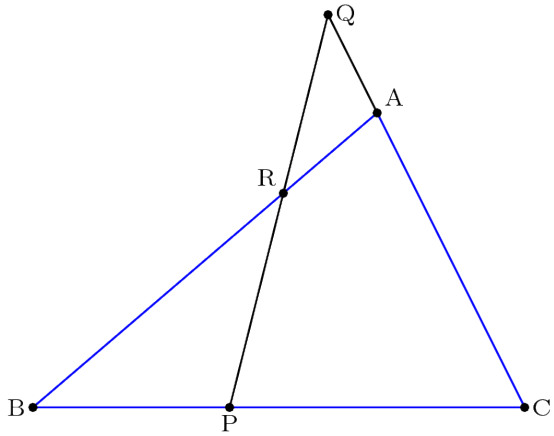
Therefore, the fly ends up in $\left(\frac{2}{5}, \frac{4}{5}\right)$.

§20 Menelaus' theorem

Theorem 20.1

If line PQ intersecting AB on $\triangle ABC$, where P is on BC, Q is on the extension of AC, and R on the intersection of PQ and AB, then

$$\frac{PB}{CP} \cdot \frac{QC}{QA} \cdot \frac{AR}{RB} = 1.$$



Example 20.2

In triangle ABC, $AB = 20$ and $AC = 11$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. The ratio of CP to PA can be expressed in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proof. First, we will find $\frac{MP}{BP}$. By Menelaus on $\triangle BDM$ and the line AC, we have

$$\frac{BC}{CD} \cdot \frac{DA}{AM} \cdot \frac{MP}{PB} = 1 \implies \frac{62MP}{11BP} = 1 \implies \frac{MP}{BP} = \frac{11}{62}.$$

This implies that $\frac{MB}{BP} = 1 - \frac{MP}{BP} = \frac{51}{62}$. Then, by Menelaus on $\triangle AMP$ and line BC, we have

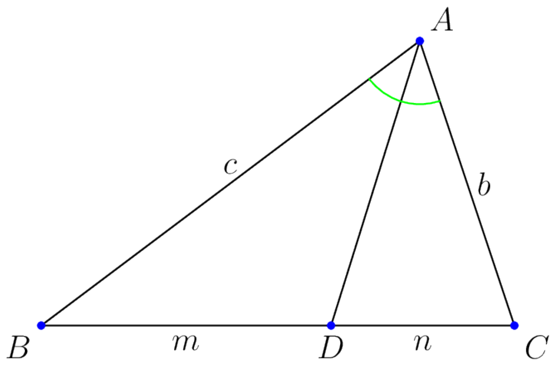
$$\frac{AD}{DM} \cdot \frac{MB}{BP} \cdot \frac{PC}{CA} = 1 \implies \frac{PC}{CA} = \frac{31}{51}.$$

Therefore, $\frac{PC}{AP} = \frac{31}{51-31} = \frac{31}{20}$. The answer is $\boxed{051}$. □

§21 Angle bisector theorem

Theorem 21.1

The Angle bisector theorem states that given triangle $\triangle ABC$ and angle bisector AD, where D is on side BC, then $\frac{c}{m} = \frac{b}{n}$. It follows that $\frac{c}{b} = \frac{m}{n}$.



Example 21.2

In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?

Proof. Denote $[\triangle ABC]$ as the area of triangle ABC and let r be the inradius. Also, as above, use the angle bisector theorem to find that $BD = 3$. There are two ways to continue from here:

1. Note that F is the incenter. Then, $\frac{AF}{FD} = \frac{[\triangle AFB]}{[\triangle BFD]} = \frac{AB*\frac{r}{2}}{BD*\frac{r}{2}} = \frac{AB}{BD} = \boxed{\textbf{(C) } 2:1}$
2. Apply the angle bisector theorem on $\triangle ABD$ to get $\frac{AF}{FD} = \frac{AB}{BD} = \frac{6}{3} = \boxed{\textbf{(C) } 2:1}$ □

Example 21.3

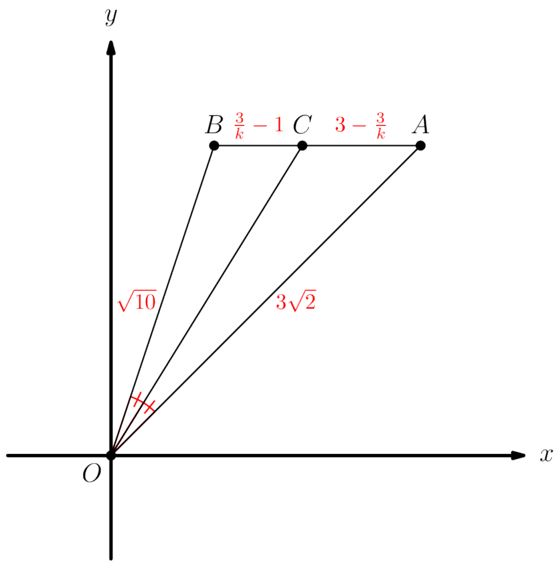
The angle bisector of the acute angle formed at the origin by the graphs of the lines $y = x$ and $y = 3x$ has equation $y = kx$. What is k?

Proof. This solution refers to the Diagram section.

Let $O = (0, 0), A = (3, 3), B = (1, 3)$, and $C = \left(\frac{3}{k}, 3\right)$. As shown below, note that $\overline{OA}, \overline{OB}$, and $\overline{OC}$ are on the lines $y = x, y = 3x$, and $y = kx$, respectively. By the Distance Formula, we have $OA = 3\sqrt{2}, OB = \sqrt{10}, AC = 3 - \frac{3}{k}$, and $BC = \frac{3}{k} - 1$.] By the Angle Bisector Theorem, we get $\frac{OA}{OB} = \frac{AC}{BC}$, or

$$\begin{aligned}
\frac{3\sqrt{2}}{\sqrt{10}} &= \frac{3 - \frac{3}{k}}{\frac{3}{k} - 1} \\
\frac{3\sqrt{2}}{\sqrt{10}} &= \frac{3k - 3}{3 - k} \\
\frac{\sqrt{2}}{\sqrt{10}} &= \frac{k - 1}{3 - k} \\
\frac{1}{5} &= \frac{(k-1)^2}{(3-k)^2} \\
5(k-1)^2 &= (3-k)^2 \\
4k^2 - 4k - 4 &= 0 \\
k^2 - k - 1 &= 0 \\
k &= \frac{1 \pm \sqrt{5}}{2}.
\end{aligned}$$

Since $k > 0$, the answer is $k = \boxed{\textbf{(A)}\ \frac{1+\sqrt{5}}{2}}$. □



Example 21.4

In triangle ABC, $AB = 13$, $BC = 14$, $AC = 15$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. Which of the following is closest to the area of the triangle ADE?

Proof. The area of ADE is $\frac{DE \cdot h}{2} = \frac{DE}{BC} \cdot \frac{BC \cdot h}{2} = \frac{DE}{BC}[ABC]$ where h is the height of triangle ABC. Using Angle Bisector Theorem, we find $\frac{13}{BE} = \frac{15}{14-BE}$, which we solve to get $BE = \frac{13}{2}$. D is the midpoint of BC so $BD = 7$. Thus we get the base of triangle ADE, DE, to be $\frac{1}{2}$ units long. We can now use Heron's Formula on ABC.

$$s = \frac{AB + BC + AC}{2} = 21$$

$$[ABC] = \sqrt{(s)(s-AB)(s-BC)(s-AC)} = \sqrt{(21)(8)(7)(6)} = 84$$

$$\frac{DE}{BC}[ABC] = \frac{\frac{1}{2}}{14} \cdot 84 = 3$$

□

§22 Practice Questions

1. Find the value of x, y in the following equation, given that $x, y \in \mathbb{R}$

$$\frac{1}{x+iy} + \frac{1}{1+2i} = 1$$

2. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

3. Show whether the following series is converging or diverging and evaluate the value to which it converges (if it does converge).

$$\sum_{n=1}^{\infty} \frac{1}{n} + \frac{1}{n+2}$$

4. Given an a by b rectangle with a fixed perimeter $P = 2(a+b)$. Show that the maximum area occurs when $a = b$.

5. What is the minimum value of

$$\frac{18}{n} + \frac{n}{2}$$

for positive values of n.

6. Find three solutions to the equation

$$4z^2 + 4\bar{z} + 1 = 0$$

where $\bar{z}$ denotes the conjugate of z.

7. Find the minimum value of

$$\frac{a}{2b} + \frac{b}{4c} + \frac{c}{8a}$$

for positive a, b, c

8. It is given that $z = 2$ and $z = 1 + 2i$ are solutions of the equation

$$z^4 - 3z^3 + az^2 + bz + c = 0$$

where a, b, c are real constants. Determine the values of a, b, c.

9. Evaluate

$$\sum_{n=1}^{38} \ln \frac{x}{x+1}$$

10. The complex number z satisfies

$$z + 1 + 8i = \mid z \mid (1 + I)$$

Show that

$$\mid z \mid^2 - 18 \mid z \mid + 65 = 0$$

($\mid z \mid$ means $\sqrt{a^2 + b^2}$ for some $z = a + ib$)

11. It is given that

$$z = \cos\theta + i\sin\theta$$

for $0 \leq z \leq 2\pi$. Show that

$$\frac{2}{1+z} = 1 - i\tan\left(\frac{\theta}{2}\right)$$

12. Let b, h denote the base and the height of some triangle whose area is 200. Find the minimum value of $b + h$

13. What is the minimum value of $x^2+y^2+z^2$, where x, y, z are positive numbers and $xyz = 1$

14. Evaluate
$$\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2}$$

15. Show that an equilateral triangle has the most area of any triangle with a fixed perimeter.

16. If x_1, x_2, x_3 are three positive numbers such that $x_1+2x_2+3x_3 = 60$. What is the smallest possible value of the sum $x_1^2+x_2^2+x_3^2$?

17. In triangle ABC,
$$2a^2+4b^2+c^2 = 4ab+2ac$$
Find the numerical value of $\cos B$

18. Demonstrate that in the case where $a_1a_2a_3...a_n = 1$, then $a_1+a_2+a_3+...+a_n \geq n$

19. Solve the equation
$$\frac{2\bar{z}(1-2i)}{5z} + \frac{i}{1+2i} = \frac{2-3i}{z}$$
giving the answer in the form of $x+iy$. $\bar{z}$ denotes the conjugate of z

20. Evaluate
$$\sum_{n=1}^{\infty} \sin^2\frac{1}{n} - \sin^2\frac{1}{n+2}$$

21. Triangle ABC has $AB = 21$, $AC = 22$ and $BC = 20$. Points D and E are located on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$ and contains the center of the inscribed circle of triangle ABC. Then $DE = m/n$, where m and n are relatively prime positive integers. Find $m+n$.

22. Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

23. Square $ABCD$ has side length 2. M is the midpoint of CD, and N is the midpoint of BC. P is on MN such that N is between M and P, and $m\angle MAN = m\angle NAP$. Compute the length of AP.

24. Let m, n be positive integers with $m < n$. Find the a closed form for the sum
$$\frac{1}{\sqrt{m}+\sqrt{m+1}} + \frac{1}{\sqrt{m+1}+\sqrt{m+2}} + \cdots + \frac{1}{\sqrt{n-1}+\sqrt{n}}$$

25. Evaluate
$$\frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + ... + \frac{1}{\sqrt{167}+\sqrt{169}}$$

26. Evaluate the following expression
$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + ... + \frac{1}{\sqrt{2016}+\sqrt{2017}}$$

27. Prove that if $a \geq b \geq 0$, then
$$\frac{(a-b)^2}{8a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{(a-b)^2}{8b}$$

28. In how many ways can the following prizes be given away to a class of twenty boys: first and second Classical, first and second Mathematical, first Science, and first French?

29. Under old hardware, a certain programme accepted passwords of the form

    eell

    where
$$e \in \{0, 2, 4, 6, 8\}, \quad l \in \{a, b, c, d, u, v, w, x, y, z\}.$$

    The hardware was changed and now the software accepts passwords of the form

    eeelll.

    How many more passwords of the latter kind are there than of the former kind?

30. A license plate is to be made according to the following provision: it has four characters, the first two characters can be any letter of the English alphabet and the last two characters can be any digit. One is allowed to repeat letters and digits. How many different license plates can be made?

31. In problem 3, how many different license plates can you make if (i) you may repeat letters but not digits?, (ii) you may repeat digits but not letters?, (iii) you may repeat neither letters nor digits? 462 Problem How many 5-lettered words can be made out of 26 letters, repetitions allowed, but not consecutive repetitions (that is, a letter may not follow itself in the same word)?

32. How many positive integers are there having $n \geq 1$ digits?

33. How many n-digits integers ($n \geq 1$) are there which are even?

34. How many n-digit nonnegative integers do not contain the digit 5 ?

35. How many n-digit numbers do not have the digit 0 ?

36. There are m different roads from town A to town B. In how many ways can Dwayne travel from town A to town B and back if (a) he may come back the way he went?, (b) he must use a different road of return?

37. How many positive divisors does $2^8 3^9 5^2$ have? What is the sum of these divisors?

38. How many factors of 2^{95} are larger than $1,000,000$?

39. How many positive divisors does 360 have? How many are even? How many are odd? How many are perfect squares?

40. How many different sums can be thrown with two dice, the faces of each die being numbered $0, 1, 3, 7, 15, 31$?

41. How many different sums can be thrown with three dice, the faces of each die being numbered $1, 4, 13, 40, 121, 364$?

42. How many two or three letter initials for people are available if at least one of the letters must be a D and one allows repetitions?

43. How many strictly positive integers have all their digits distinct?

44. The Morse code consists of points and dashes. How many letters can be in the Morse code if no letter contains more than four signs, but all must have at least one?

45. An $n \times n \times n$ wooden cube is painted blue and then cut into $n^3 1 \times 1 \times 1$ cubes. How many cubes (a) are painted on exactly three sides, (b) are painted in exactly two sides, (c) are painted in exactly one side, (d) are not painted?

46. How many even integers between 4000 and 7000 have four different digits? 494 Problem (AHSME 1998) Call a 7-digit telephone number $d_1 d_2 d_3 - d_4 d_5 d_6 d_7$ memorable if the prefix sequence $d_1 d_2 d_3$ is exactly the same as either of the sequences $d_4 d_5 d_6$ or $d_5 d_6 d_7$ or possibly both. Assuming that each d_i can be any of the ten decimal digits $0, 1, 2, \ldots, 9$, find the number of different memorable telephone numbers.

47. Three-digit numbers are made using the digits $\{1, 3, 7, 8, 9\}$.

48. How many of these integers are there?

49. How many are even?

50. How many are palindromes?

51. How many are divisible by 3 ?

52. (AHSME 1989) Five people are sitting at a round table. Let $f \geq 0$ be the number of people sitting next to at least one female, and let $m \geq 0$ be the number of people sitting next to at least one male. Find the number of possible ordered pairs (f, m)

53. How many integers less than 10000 can be made with the eight digits $0, 1, 2, 3, 4, 5, 6, 7$?

54. In how many ways can one arrange the numbers $21, 31, 41, 51, 61, 71$, and 81 such that the sum of every four consecutive numbers is divisible by 3 ?

55. Let S be the set of all natural numbers whose digits are chosen from the set $\{1, 3, 5, 7\}$ such that no digits are repeated. Find the sum of the elements of S.

56. Show that

$$1+2+3+\cdots+\left(n^2-1\right)+n^2=\frac{n^2\left(n^2+1\right)}{2}.$$

57. Show that

$$1+3+5+\cdots+2n-1=n^2.$$

58. Sum the series

$$20+20\frac{1}{5}+20\frac{2}{5}+\cdots+40.$$

59. Show that

$$\frac{1}{1996}+\frac{2}{1996}+\frac{3}{1996}+\cdots+\frac{1995}{1996}$$

is an integer.

60. Let $T_n = 1+2+3+\cdots+n$ and

$$P_n = \frac{T_2}{T_2-1}\cdot\frac{T_3}{T_3-1}\cdot\frac{T_4}{T_4-1}\cdots\frac{T_n}{T_n-1}.$$

Find P_{991}.

61. Given that

$$\frac{1}{a+b},\frac{1}{b+c},\frac{1}{c+a}$$

are consecutive terms in an arithmetic progression, prove that

$$b^2,a^2,c^2$$

are also consecutive terms in an arithmetic progression. 360 Problem Consider the following table:

$$\begin{gathered}1=1\\ 2+3+4=1+8\\ 5+6+7+8+9=8+27\\ 10+11+12+13+14+15+16=27+64\end{gathered}$$

Conjecture the law of formation and prove your answer.

62. The odd natural numbers are arranged as follows:

$$\begin{gathered}(1)\\ (3,5)\\ (7,9,11)\\ (13,15,17,19)\\ (21,23,25,27,29)\end{gathered}$$

Find the sum of the nth row.

63. Sum

$$1000^2-999^2+998^2-997^2+\cdots+4^4-3^2+2^2-1^1.$$

64. The first term of an arithmetic progression is 14 and its 100 th term is -16. Find (i) its 30th term and (ii) the sum of all the terms from the first to the 100 th.